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ORIGINAL RESEARCH

Lipschitz retractions on symmetric products of trees

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Abstract Given a continuum X and a positive integer n, $F_n(X)$ denotes the hyperspace of non-empty subsets of X with at most n elements, endowed with the Hausdorff metric. In this article, given X a simple m-od, we prove that $F_{n-1}(X)$ is a $(6n+1)$ - Lipschitz retract of $F_n(X)$ for every $n \geq 2$, and that $F_{n-1}(X)$ is a 4-Lipschitz retract of $F_n(X)$ for X a tree and $n = 2, 3$.

Keywords Symmetric products · Retractions · Lipschitz maps · Trees and m-ods

Mathematics Subject Classification 54B20 · 54C15 · 54E40

1 Introduction

A *continuum* is a non degenerated compact connected metric space. Given $n \in \mathbb{N}$, the n-fold symmetric product of a continuum X, $F_n(X)$, is defined as the space of non-empty subsets of cardinality at most n, endowed with the Hausdorff metric. The symmetric products were introduced by K. Borsuk and S. Ulam in [4].

A map $f : (X, d_X) \to (Y, d_Y)$ is called Lipschitz if there is a real number $L > 0$, such that $d_Y(f(a), f(b)) \leq L d_X(a, b)$. A subset Y of a continuum X is a Lipschitz retract of X if there exists a Lipschitz map $r : X \to Y$ that fixes Y pointwise. Notice that $F_1(X) \subset F_2(X) \subset F_3(X) \subset \cdots$. L. V. Kovalev proposed the following interesting problem in [7, Problem 4.1, p. 806]:

Problem 1 Characterize the metric spaces X such that $F_k(X)$ is a Lipschitz retract of $F_n(X)$ whenever $k < n$.

In general there are no retractions $r : F_n(X) \to F_k(X)$ (with $1 \leq k < n$), for example if X is the circle $\mathbb{S}^1$ it is known that $F_2(X)$ is homeomorphic to the Möbius band, see [2, p. 887], where $F_1(\mathbb{S}^1)$ could be identified with its border, also R. Bott in [5] proved that $F_3(X)$ is homeomorphic to the 3-sphere, $\mathbb{S}^3$. However, $F_1(\mathbb{S}^1)$

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is embedded in the form of a trefoil knot in $F_3(\mathbb{S}^1)$ as shown by J. Mostovoy in [10]. In a positive way, L. V. Kovalev in [7, Lemma 4.2, p. 806] showed that for any connected and non-empty subset X of the real line, there exists $r : F_n(X) \to F_k(X)$ a Lipschitz retraction. In [3] and [6], Lipschitz retractions are studied in symmetric products of Hadamard spaces and [8] on Hilbert spaces of finite and infinite dimensions. Recently, in [1] Lipschitz retractions are studied in symmetric products of normed spaces, in particular E. Akofor in [1, Theorem 3.18, p.7] proved that for a normed space X there exists a 731-Lipschitz retraction $r : F_3(X) \to F_2(X)$.

In Section 3 of this article we prove that for every $n \geq 2$, $F_{n-1}(X)$ is a $(6n+1)$-Lipschitz retract of $F_n(X)$ for X a simple m-od, and in Section 4 we prove that if X is a tree then $F_{n-1}(X)$ is a 4-Lipschitz retract of $F_n(X)$ for $n = 2, 3$. At the end of this paper, some problems are proposed that remain open.

2 Preliminaries

Given a continuum X with metric d, $a \in X$, $A \subset X$ and $\varepsilon > 0$. $B_d(a, \varepsilon)$ denotes the open ball of radio ε with center in a and we define

$$B_d(A, \varepsilon) = \bigcup_{a \in A} B_d(a, \varepsilon),$$
$$\operatorname{dist}(a, A) = \min\{d(a, b) : b \in A\}.$$

Let $n \in \mathbb{N}$. We consider

$$F_n(X) = \{A \subset X : 1 \leq |A| \leq n\}$$

with the Hausdorff metric given by

$$\begin{aligned} H(A,B) &= \inf\{\varepsilon > 0 : A \subset B_d(B, \varepsilon) \text{ and } B \subset B_d(A, \varepsilon)\} \\ &= \max\{\max_{a \in A} \operatorname{dist}(a, B), \max_{b \in B} \operatorname{dist}(b, A)\} \end{aligned}$$

for any $A, B \in F_n(X)$. A *finite graph* is a continuum that can be written as the union of a finite number of arcs, which we will call edges, say $e_1, e_2, \ldots, e_n$, such that e_i intersects e_j only on one or both end points, to $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

A *tree* is a finite graph without subspaces homeomorphic to the circle $\mathbb{S}^1$. Given $m \in \mathbb{N}$, a *simple m-od* is a tree with m edges $e_1, e_2, \ldots, e_m$, such that $e_i \cap e_j = v$ for any $i \neq j$ and v a fixed point. Let $A \in F_n(X)$. If $|A| = n$, we define $\delta(A) = \min\{d(a, b) : a, b \in A, a \neq b\}$ otherwise $\delta(A) = 0$. The following lemma is easy to verify.

Lemma 2 *Let (X, d) be a metric space and $A, B \in F_n(X)$. If $a \in A$ then there is $b \in B$ such that $d(a, b) \leq H(A, B)$. Moreover, $dist(a, B) \leq H(A, B)$.*

Note that the previous lemma is valid if A and B are elements of the hyperspace of closed subsets of X. The next lemma shows that the function $\delta : F_n(X) \to [0, \infty)$ is Lipschitz.

Lemma 3 *Let (X, d) be a metric space. If $A, B \in F_n(X)$, then $|\delta(A) - \delta(B)| \leq 2H(A, B)$*

Proof To prove this lemma we consider two cases:

Case I. If $\delta(A) = \delta(B)$, then $|\delta(A) - \delta(B)| = 0 \leq 2H(A, B)$.

Case II. If $\delta(A) \neq \delta(B)$. Suppose that $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. Without loss of generality we can assume that $\delta(A) > \delta(B)$. Let b_1 and b_2 such that $\delta(B) = d(b_1, b_2)$, in case $\delta(B) = 0$, we consider $b_1 = b_2$. By Lemma 2 we have that there exists $a_1, a_2 \in A$ such that

$$d(b_1, a_1) \leq H(A, B) \text{ and } d(b_2, a_2) \leq H(A, B). \tag{1}$$

For the other hand

$$d(a_1, a_2) \leq d(a_1, b_1) + d(b_1, b_2) + d(b_2, a_2). \tag{2}$$

So, by (1) and (2) we obtain,

$$d(a_1, a_2) - d(b_1, b_2) \le 2H(A, B). \tag{3}$$

Similarly,

$$d(b_1, b_2) - d(a_1, a_2) \le 2H(A, B). \tag{4}$$

Therefore by (3) and (4),

$$\mid \delta(A) - \delta(B) \mid \;\le\; \mid d(b_1, b_2) - d(a_1, a_2) \mid \;\le 2H(A, B).$$

The lemma is demonstrated. ☐

3 Simple *m*-ods

In this section we prove that for any simple m-od it is satisfied that its k-th symmetric product is a Lipschitz retract of its n-th symmetric product for $1 \le k < n$.

Let $m \in \mathbb{N}$, in this section we consider a simple m-od, X, defined in the following way

$$X = \bigcup_{i=1}^{m} T_i$$

where $T_1 = \{(x_1, 0, \ldots, 0) \in \mathbb{R}^m : 0 \le x_1 \le 1\}$ and $T_k = \{(0, \ldots, x_k, \ldots, 0) \in \mathbb{R}^m : -1 \le x_k < 0\}$ for $k = 2, \ldots, m$. We consider X with the given metric by $d(x, y) = \sum_{i=1}^{m} |x_i - y_i|$ for any $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in X$.

For each $i = 1, \ldots, m$, $\pi_i : \mathbb{R}^m \to \mathbb{R}$ denotes the *i-th projection* given by $\pi_i((x_1, \ldots, x_m)) = x_i$. Given $a, b \in T_i$ we say that $a \le b$ if and only if $\pi_i(a) \le \pi_i(b)$. Given $A \in F_n(X)$, for each $i = 1, \ldots, m$; suppose that $A \cap T_i = \{t_1^i, \ldots, t_{r_i}^i\}$ with $t_{j-1}^i < t_j^i$ for $j = 2, \ldots, r_i$.

For each $a \in A$ we define a' in the following way:

- If $a \in T_1$, then $a = t_j^1$ for some $j \in \{1, \ldots, r_1\}$; thus $a' = \left(t_j^1\right)' = \left(\max\left\{0, \pi_1(t_j^1) - \left(\sum_{l=2}^{m} r_l + j\right)\delta(A)\right\}, 0, \ldots, 0\right)$.
- If $a \in T_i$ for $i \ne 1$ then $a = t_j^i$ for some $j \in \{1, \ldots, r_i\}$, thus $a' = \left(t_j^i\right)' = \left(0, \ldots, \min\left\{0, \pi_i(t_j^i) + (n-j)\delta(A)\right\}, \ldots, 0\right)$.

Proposition 4 *If $a \in T_1$ then $a' \in T_1$ and $a' \le a$.*

Proof Since $a \in T_1$ then $a = t_j^1$ for some $j \in \{1, \ldots, r_1\}$. For the first part, notice that

$$0 \le \pi_1(a') = \max\left\{0, \pi_1\left(t_j^1\right) - \left(\sum_{l=2}^{m} r_l + j\right)\delta(A)\right\} \le \pi_1(t_j^1) = \pi_1(a) \le 1,$$

so $a' \in T_1$. For the second part, since $0 \le \pi_1(a)$ and

$$\pi_1(a) - \left(\sum_{l=2}^{m} r_l + j\right)\delta(A) \le \pi_1(a),$$

then $\max\left\{0, \pi_1(t_j^1) - (\sum_{l=2}^{m} r_l + j)\delta(A)\right\} \le \pi_1(a)$. Therefore $a' \le a$. ☐

Proposition 5 *If $a \in T_i$ for some $i \in \{2, \ldots, m\}$ then $a' \in T_i \cup \{\overline{0}\}$ and $a \le a'$.*

Proof As $a \in T_i$ then $a = t_j^i$ for some $j = 1, \ldots, r_i$. Thus $-1 \le \pi_i(t_j^i) < 0$ and since $(n-j)\delta(A) \ge 0$, we have that $\pi_i(t_j^i) \le \pi_i(t_j^i) + (n-j)\delta(A)$ and since $\pi_i(t_j^i) < 0$ we obtain

$$-1 \le \pi_i(t_j^i) \le \pi_i\left((t_j^i)'\right) = \min\left\{0, \pi_i(t_j^i) + (n-j)\delta(A)\right\} \le 0$$

that is to say, $(0,\ldots,-1,\ldots,0)\leq a\leq a'\leq\overline{0}$. Therefore $a\leq a'$ and $a'\in T_i\cup\{\overline{0}\}$. $\square$

Proposition 6 *If $t^i_{j-1}, t^i_j\in T_i$ for some $i\in\{1,\ldots,m\}$ and for some $j\in\{2,\ldots,r_i\}$, then $\left(t^i_{j-1}\right)'\leq\left(t^i_j\right)'$.*

Proof We consider two cases:

- Case I. Suppose that $i=1$. By definition of $\delta(A)$ we obtain

$$\pi_1(t^i_{j-1})+\delta(A)\leq\pi_1(t^i_j)$$

thus,

$$\begin{aligned}\pi_1(t^i_{j-1})-\left(\sum_{l=2}^m r_l+(j-1)\right)\delta(A)&=\pi_1(t^i_{j-1})+\delta(A)-\left(\sum_{l=2}^m r_l+j\right)\delta(A)\\&\leq\pi_1(t^i_j)-\left(\sum_{l=2}^m r_l+j\right)\delta(A).\end{aligned}$$

Then,

$$\begin{aligned}\pi_1((t^i_{j-1})')&=\max\left\{0,\pi_1(t^i_{j-1})-\left(\sum_{l=2}^m r_l+(j-1)\right)\delta(A)\right\}\\&\leq\max\left\{0,\pi_1(t^i_j)-\left(\sum_{l=2}^m r_l+j\right)\delta(A)\right\}=\pi_1((t^i_j)').\end{aligned}$$

Therefore $(t^i_{j-1})'\leq(t^i_j)'$.

- Case II. Suppose that $i\in\{2,\ldots,m\}$. By definition of $\delta(A)$ we have that

$$\pi_1(t^i_{j-1})+\delta(A)\leq\pi_1(t^i_j),$$

then

$$\begin{aligned}\pi_1(t^i_{j-1})+(n-j+1)\delta(A)&=\pi_1(t^i_{j-1})+\delta(A)+(n-j)\delta(A)\\&\leq\pi(t_j)+(n-j)\delta(A).\end{aligned}$$

Therefore $(t^i_{j-1})'\leq(t^i_j)'$. $\square$

Proposition 7 *If $c,d\in A$ are such that $\delta(A)=d(c,d)$, then $c'=d'$.*

Proof We consider the following three cases:
Case I. If $c,d\in T_1$ then $c=t^1_{j-1}$ and $d=t^1_j$ for some $j\in\{2,\ldots,r_1\}$.
Since $\delta(A)=d(c,d)$, we have that $\pi_1(t^1_{j-1})+\delta(A)=\pi_1(t^1_j)$, thus

$$\pi_1(t^1_{j-1})+\delta(A)-\left(\sum_{l=2}^m r_l+j\right)\delta(A)=\pi_1(t^1_j)-\left(\sum_{l=2}^m r_l+j\right)\delta(A)$$

as $\delta(A)-\left(\sum_{l=2}^m r_l+j\right)\delta(A)=-\left(\sum_{l=2}^m r_l+(j-1)\right)\delta(A)$, we obtain

$$\pi_1(t^1_{j-1})-\left(\sum_{l=2}^m r_l+(j-1)\right)\delta(A)=\pi_1(t^1_j)-\left(\sum_{l=2}^m r_l+j\right)\delta(A).$$

Thus

$$\begin{aligned}\pi_1((t^i_{j-1})') &= \max\left\{0, \pi_1(t^1_{j-1}) - \left(\sum_{l=2}^{m} r_l + (j-1)\right)\delta(A)\right\}\\ &= \max\left\{0, \pi_1(t^1_{j}) - \left(\sum_{l=2}^{m} r_l + j\right)\delta(A)\right\} = \pi_1((t^i_j)').\end{aligned}$$

Therefore $c' = (t^i_{j-1})' = (t^i_j)' = d'$.

Case II. If $c, d \in T_i$ for some $i \in \{2, \ldots, m\}$, then $c = t^i_{j-1}$ and $d = t^i_j$ for some $j \in \{2, \ldots, r_i\}$. Since $\delta(A) = d(c, d)$, we have that $\pi_i(t^i_{j-1}) + \delta(A) = \pi_i(t^i_j)$, then

$$\pi_i(t^i_{j-1}) + \delta(A) + (n-j)\delta(A) = \pi_i(t^i_j) + (n-j)\delta(A).$$

As $\delta(A) + (n-j)\delta(A) = (n-j+1)\delta(A)$ we obtain

$$\pi_i(t^i_{j-1}) + (n-(j-1))\delta(A) = \pi_i(t^i_j) + (n-j)\delta(A).$$

Thus

$$\begin{aligned}\pi_i((t^i_{j-1})') &= \max\left\{0, \pi_1(t^i_{j-1}) - (n-(j-1))\delta(A)\right\}\\ &= \max\left\{0, \pi_1(t^i_{j}) - (n-j)\delta(A)\right\} = \pi_1((t^i_j)')..\end{aligned}$$

And in this case it is also concluded that $c' = (t^i_{j-1})' = (t^i_j)' = d'$.

Case III. $c \in T_u$, $d \in T_v$ with $u \neq v$ and $u, v \in \{1, \ldots m\}$. We consider the following two subcases:

Subcase I. $c \in T_1$ or $d \in T_1$. Without loss of generality suppose that $c \in T_1$, then $c = t^1_j$ for some $j \in \{1, \ldots, r_1\}$. Notice that

$$\delta(A) = d(c, d) = d(\pi_1(c), 0) + d(0, \pi_v(d)), \tag{5}$$

thus $\pi_1(c) - \delta(A) \leq \pi_1(c) - d(\pi_1(c), 0) = 0$.

As $\pi_1(c) - \left(\sum_{l=2}^{m} r^1_l + j\right)\delta(A) \leq \pi_1(c) - \delta(A)$,

then $\pi_1(c) - \left(\sum_{l=2}^{m} r^1_l + j\right)\delta(A) \leq 0$. Hence

$$\max\left\{0, \pi_1(c) - \left(\sum_{l=2}^{m} r^1_l + j\right)\delta(A)\right\} = 0.$$

Thus $c' = \overline{0}$.

Subcase II. $c \in T_i$ or $d \in T_i$ for some $i \in \{2, \ldots, m\}$. Without loss of generality suppose that $d \in T_i$, thus $d = t^i_j$ for some $j = 1, \ldots, r_i$. By (5) we have that

$$\begin{aligned}0 &= \pi_i(d) + d(\pi_i(d), 0)\\ &\leq \pi_i(d) + d(\pi_i(d), 0) + d(0, \pi_i(c))\\ &= \pi_i(d) + \delta(A) \leq \pi_i(d) + (n-j)\delta(A).\end{aligned}$$

Thus,

$$\min\{0, \pi_i(d) + (n-j)\delta(A)\} = 0$$

that is, $d' = \overline{0}$.

Therefore $c' = d'$. ☐

Lemma 8 *If $A = \{a_1, \ldots, a_m\} \in F_n(X)$ then*

$$\max\left\{d(a_k, a'_k) : k \in \{1, \ldots, m\}\right\} \leq n\delta(A).$$

Proof To prove this lemma we require the following cases: **Case I.** If $|A| < n$. By definition we have that $\delta(A) = 0$ and $a_k = a'_k$ for each $k \in \{1, \ldots, m\}$. So

$$\max\left\{d(a_k, a'_k) : k \in \{1, \ldots, m\}\right\} = 0 \leq n\delta(A).$$

Case II. If $|A| = n$. We consider the following two subcases:

- If $a_k \in T_1$ then $a_k = t^1_j$ for some $j \in \{1, \ldots, r_1\}$, by Proposition 4 we have that $a'_k \leq a_k$, and since $\sum_{l=2}^{m} r_l + j \leq n$

$$\begin{aligned} d(a_k, a'_k) &\leq \left| \pi_1(t^1_j) - \left(\pi_1(t^1_j) - \left(\sum_{l=2}^{m} r_l + j \right) \delta(A) \right) \right| \\ &= \left| \left(\sum_{l=2}^{m} r_l + j \right) \delta(A) \right| \\ &= \left(\sum_{l=2}^{m} r_l + j \right) \delta(A) \\ &\leq n\delta(A). \end{aligned}$$

- If $a_k \in T_i$, for some $i \in \{2, \ldots, m\}$, then $a_k = t^i_j$ for some $j \in \{1, \ldots, r_j\}$. Hence,

$$\begin{aligned} d(a_k, a'_k) &\leq |\pi_i(t^i_j) - \left(\pi_i(t^i_j) + (n-j)\delta(A) \right)| \\ &= (n-j)\delta(A) \\ &\leq n\delta(A). \end{aligned}$$

Thus the lemma is demonstrated. □

Now, for $n \geq 2$ we define the map $r : F_n(X) \longrightarrow F_{n-1}(X)$ given by:

$$r(A) = \begin{cases} \{a'_1, \ldots, a'_n\} & \text{if } |A| = n, \\ \\ A & \text{if } |A| < n. \end{cases} \tag{6}$$

By the propositions 4, 5, 6 and 7 we have that r is well defined. Next we will show an important property of the map r that will be very useful in order to prove the main theorem of this section.

Lemma 9 *If* $A = \{a_1, \ldots, a_m\} \in F_n(X)$ *then*

$$H(A, r(A)) \leq \max\left\{ d(a_k, a'_k) : k \in \{1, \ldots, m\} \right\}.$$

Proof For the proof of this lemma we consider two cases: **Case I.** If $|A| < n$, by definition we have that $r(A) = A$, so

$$H(A, r(A)) = 0 \leq \max\left\{ d(a_k, a'_k) : k \in \{1, \ldots, m\} \right\}.$$

Case II. If $|A| = n$.

Since

$$H(A, r(A)) = \max\{\max_{a \in A} \operatorname{dist}(a, r(A)), \max_{a' \in r(A)} \operatorname{dist}(a', A)\},$$

it is enough to demonstrate the following two things:

i) $\max_{a \in A} \operatorname{dist}(a, r(A)) \leq \max\left\{ d(a_k, a'_k) : k \in \{1, \ldots, m\} \right\}$, and
ii) $\max_{a' \in r(A)} \operatorname{dist}(a', A) \leq \max\left\{ d(a_k, a'_k) : k \in \{1, \ldots, m\} \right\}$.

To show i), given $a_j \in A$ with $j \in \{1, \ldots, n\}$, we have that

$$\begin{aligned}\operatorname{dist}(a_j, r(A)) &= \min\{d(a_j, a_i') | a_i' \in r(A)\}\\ &\leq d(a_j, a_j')\\ &\leq \max\{d(a_k, a_k') | k = 1, \ldots, n\}.\end{aligned}$$

Similarly to prove ii), let $a_j' \in r(A)$ with $j \in \{1, \ldots, n\}$, then we have that

$$\begin{aligned}\operatorname{dist}(a_j', A) &= \min\Big\{d(a_j', a) | a \in A\Big\}\\ &\leq d(a_j, a_j')\\ &\leq \max\{d(a_k, a_k') | k = 1, \ldots, n\}.\end{aligned}$$

Therefore, if $|A| = n$ we obtain

$$H(A, r(A)) \leq \max\{d(a_k, a_k') | k = 1, \ldots, n\}.$$

□

Theorem 10 *Let $n \geq 2$. If X is a simple m-od then $F_{n-1}(X)$ is a Lipschitz retract of $F_n(X)$.*

Proof It suffices to prove that the retraction r given in (6) is Lipschitz, for which we consider two cases:
Case I. $A = B$.

Clearly in this case the Lipschitz inequality is satisfied since

$$H(r(A), r(B)) = 0 = H(A, B).$$

Case II. $A \neq B$.

In this case, consider the following two subcases:

Subcase I. $\delta(A) \leq 2H(A, B)$. By Lemma 3 we have that

$$\begin{aligned}\delta(B) &\leq |\delta(B) - \delta(A)| + \delta(A) \leq 2H(A, B) + \delta(A)\\ &\leq 2H(A, B) + 2H(A, B) = 4H(A, B).\end{aligned}$$

Thus, also applying lemmas 8 and 9 we obtain

$$\begin{aligned}H(r(A), r(B)) &\leq H(r(A), A) + H(A, B) + H(B, r(B))\\ &\leq n\delta(A) + H(A, B) + n\delta(B)\\ &\leq 2nH(A, B) + H(A, B) + 4nH(A, B)\\ &= (6n + 1)H(A, B).\end{aligned}$$

Subcase II. $\delta(A) > 2H(A, B)$.

Under these circumstances notice that $\delta(B) > 0$, otherwise by Lemma 3 we have that $\delta(A) = |\delta(A) - \delta(B)| \leq 2H(A, B)$, which is a contradiction. From the above $|A| = n = |B|$. Thus, we suppose that $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$ and let $\varepsilon = H(A, B)$.

Claim a) $B \cap B_\varepsilon(a_k) \neq \emptyset$, for each $k \in \{1, \ldots, n\}$. Indeed, by Lemma 2 we have that there exists $b \in B$ such that $d(a_k, b) \leq H(A, B)$ and thus $b \in B \cap B_\varepsilon(a_k)$.

Claim b) Given $j, k \in \{1, \ldots, n\}$ with $k \neq j$, $B_\varepsilon(a_k) \cap B_\varepsilon(a_j) = \emptyset$. Indeed, if $p \in B_\varepsilon(a_k) \cap B_\varepsilon(a_j)$, then $d(a_k, p) \leq H(A, B)$ and $d(a_j, p) \leq H(A, B)$ and so we have that $2H(A, B) < \delta(A) \leq d(a_k, a_j) \leq d(a_k, p) + d(p, a_j) \leq 2H(A, B)$, which is not possible. By claims a) and b) for each $k \in \{1, \ldots, n\}$, there is a unique $s(k)$ such that $b_{s(k)} \in B_\varepsilon(a_k)$. Also a_k and $b_{s(k)}$ satisfy the following claim.

Claim c) $d(a_k', b_{s(k)}') \leq (2n + 1)H(A, B)$. For the proof of this claim we consider the following four cases:

Case i) $a_k \in T_i$ for some $i \in \{2, \ldots, m\}$ and $b_{s(k)} \in T_1$. By propositions 4 and 5 we have that $a_k \leq a_k'$ and $b_{s(k)}' \leq b_{s(k)}$, so $d(a_k', b_{s(k)}') \leq d(a_k, b_{s(k)})$ and since $b_{s(k)} \in B_\varepsilon(a_k)$, then

$$d(a_k', b_{s(k)}') \leq d(a_k, b_{s(k)}) \leq H(A, B) < (2n + 1)H(A, B).$$

Case ii) $a_k, b_{s(k)} \in T_i$ for some $i \in \{2, \ldots, m\}$.

In this case we have that $a_k = t_j^i$ and $b_{s(k)} = s_j^i$. Let $p = \pi_i(t_j^i) + (n - j)\delta(A)$ and $q = \pi_i(s_j^i) + (n - j)\delta(B)$. Thus, $\pi_i(a_k') = \min\{0, p\} = \frac{p - |p|}{2}$ and $\pi_i(b_{s(k)}') = \min\{0, q\} = \frac{q - |q|}{2}$. Then,

$$d(a'_k, b'_{s(k)}) = \left|\frac{(p-|p|)-(q-|q|)}{2}\right|,$$

so,

$$2d(a'_k, b'_{s(k)}) = \Big|(p-q)+(|q|-|p|)\Big| \leq |p-q| + ||q|-|p||.$$

Since $||q|-|p|| \leq |p-q|$, we have $2d(a'_k, b'_{s(k)}) \leq 2|p-q|$. So, by Lema 3 we have that

$$\begin{aligned} d(a'_k, b'_{s(k)}) &\leq |\pi_i(t^i_j) + (n-j)\delta(A) - (\pi_i(s^i_j) + (n-j)\delta(B))| \\ &\leq |\pi_i(t^i_j) - (\pi_i(s^i_j)| + (n-j)|\delta(A) - \delta(B)| \\ &\leq H(A,B) + 2nH(A,B) \\ &= (2n+1)H(A,B). \end{aligned}$$

Case iii) $a_k, b_{s(k)} \in T_1$.

In this case we have that $a_k = t^1_j$ and $b_{s(k)} = s^1_j$.

Let $p = \pi_1(t^1_j) - \left(\sum_{l=2}^{m} r_l + j\right)\delta(A)$ and $q = \pi_1(s^1_j) - \left(\sum_{l=2}^{m} r_l + j\right)\delta(B)$.

Hence, $\pi_i(a'_k) = \max\{0, p\} = \frac{p+|p|}{2}$ and $\pi_i(b'_{s(k)}) = \max\{0, q\} = \frac{q+|q|}{2}$.

Continuing as in Case ii) we obtain $d(a'_k, b'_{s(k)}) \leq (2n+1)H(A,B)$.

Case iv) $a_k \in T_u$ and $b_{s(k)} \in T_v$ with $u, v \in \{2, \ldots, m\}$ and $u \neq v$. By Proposition 4 we have that $a_k \leq a'_k$ and $b_{s(k)} \leq b'_{s(k)}$, thus

$$d(a'_k, b'_{s(k)}) \leq d(a_k, b_{s(k)}) \leq H(A,B) < (2n+1)H(A,B).$$

This concludes the proof of Claim c).

Now remember that

$$H(r(A), r(B)) = \max\left\{\max_{a' \in r(A)} \operatorname{dist}(a', r(B)), \max_{b' \in r(B)} \operatorname{dist}(b', r(A))\right\}.$$

Without loss of generality, suppose that

$$H(r(A), r(B)) = \max_{a' \in r(A)} \operatorname{dist}(a', r(B)).$$

Then for some $j \in \{1, \ldots, n\}$,

$$\begin{aligned} H(r(A), r(B)) &= \operatorname{dist}(a'_j, r(B)) \\ &\leq d(a'_j, b'_{s(j)}) \\ &\leq \max_k d(a'_k, b'_{s(k)}) \\ &\leq (2n+1)H(A,B) \\ &< (6n+1)H(A,B). \end{aligned}$$

Therefore in any case we have that

$$H(r(A), r(B)) \leq (6n+1)H(A,B).$$

That is, r is $(6n+1)-$ Lipschitz. □

By [9, Proposición 2.1.3, p. 10] we have the following corollary

Corollary 11 *Let $n \geq 2$. If X is a simple m-od, then for all $1 \leq k < n$, $F_k(X)$ is a L-Lipschitz retract of $F_n(X)$, where $L = (6n+1)(6(n-1)+1)\cdots(6(k+1))+1)$.*

4 Trees

Let X be a tree, let's assume that the metric d in X is the arc length metric. Also, suppose that each edge e_i of X has length one. Given $a, b \in X$, $\overrightarrow{ab}$ denotes the arc that goes from a to b. Given $0 \leq \lambda < d(a,b)$, we define a^b_λ as the only point in the arc $\overrightarrow{ab}$ such that $d(a, a^b_\lambda) = \lambda$ and we define the midpoint of $\{a,b\}$, denoted by $PM(\{a,b\})$, as the only point in the arc $\overrightarrow{ab}$ such that $d(a, PM(\{a,b\})) = d(PM(\{a,b\}), b)$.

Remark 12 Let $a, b, c \in X$. Then $c \in \overrightarrow{ab}$ if and only if $d(a,b) = d(a,c) + d(c,b)$.

Proposition 13 *Let $a, b, v \in X$ and $0 \leq \lambda \leq d(b,v)$. If $b \in \overrightarrow{av}$ then $d(a^v_\lambda, b^v_\lambda) = d(a,b)$.*

Proof We consider two cases:

Case I. $b \in \overrightarrow{a^v_\lambda v}$. By Remark 12 $d(a^v_\lambda, b^v_\lambda) = d(a^v_\lambda, b) + d(b, b^v_\lambda) = d(a,b) - \lambda + \lambda = d(a,b)$.

Case II. $a^v_\lambda \in \overrightarrow{bb^v_\lambda}$. Since $d(a, b^v_\lambda) = d(a,b) + \lambda$, then $d(a^v_\lambda, b^v_\lambda) = d(a, b^v_\lambda) - d(a, a^v_\lambda) = d(a, b^v_\lambda) - \lambda = d(a,b)$. □

Proposition 14 *Let $a, b, v \in X$, $0 \leq \lambda \leq d(a,v)$ and $0 \leq \kappa \leq d(b,v)$. If $b \in \overrightarrow{av}$, then $d(a^v_\lambda, b^v_\kappa) \leq d(a,b) + |\lambda - \kappa|$.*

Proof Notice that if $\lambda = \kappa$, then Proposition 13 give the equality. Now if $\lambda \neq \kappa$, let's consider the following two cases:

Case I. $b \in \overrightarrow{a^v_\lambda v}$.

By Remark 12, $d(a^v_\lambda, b^v_\kappa) = d(a^v_\lambda, b) + d(b, b^b_\kappa) = d(a,b) - \lambda + \kappa \leq d(a,b) + |\lambda - \kappa|$.

Case II. $a^v_\lambda \in \overrightarrow{bv}$.

i) If $a^v_\lambda \in \overrightarrow{b^v_\kappa v}$, then $d(a^v_\lambda, b^v_\kappa) = d(a^v_\lambda, a) - d(a,b) - d(b, b_\kappa) = -d(a,b) + \lambda - \kappa \leq d(a,b) + |\lambda - \kappa|$.

ii) If $b^v_\kappa \in \overrightarrow{a^v_\lambda v}$, then $d(a^v_\lambda, b^v_\kappa) = d(a, b^v_\kappa) - d(a, a^v_\lambda) = d(a,b) + d(b, b^v_\kappa) - d(a, a^v_\lambda) \leq d(a,b) + |\lambda - \kappa|$.

□

Proposition 15 *Let $a, b, v \in X$, $0 \leq \lambda \leq d(a,v)$ and $0 \leq \kappa \leq d(b,v)$. If $v \in \overrightarrow{ab}$, then $d(a^v_\lambda, b^v_\kappa) \leq d(a,b)$.*

Proof By Remark 12 $d(a,b) = d(a, a^v_\lambda) + d(a^v_\lambda, b^v_\kappa) + d(b^v_\kappa, b) = \lambda + d(a^v_\lambda, b^v_\kappa) + \kappa$. So $d(a^v_\lambda, b^v_\kappa) = d(a,b) - (\lambda + \kappa) \leq d(a,b)$. □

Proposition 16 *Let $a, b, v \in X$, $0 \leq \lambda \leq \min\{d(a,v), d(b,v)\}$. If $a \notin \overrightarrow{bv}$ and $b \notin \overrightarrow{av}$, then $d(a^v_\lambda, b^v_\lambda) \leq d(a,b)$.*

Proof Note that if $\lambda = 0$ then the equality follows. Suppose that $\lambda > 0$, let $z \in \overrightarrow{av} \cap \overrightarrow{bv}$ such that if $w \in \overrightarrow{av} \cap \overrightarrow{bv}$ then $\overrightarrow{wv} \subset \overrightarrow{zv}$. We consider the following three cases:

Case I. $a^v_\lambda \in \overrightarrow{bv}$ and $b^v_\lambda \in \overrightarrow{av}$.

Without loss of generality, suppose that $d(a,z) \leq d(b,z)$. Let $a' \in \overrightarrow{bv}$ such that $a^v_\lambda = a'^v_\lambda$. Note that $d(a', b) < d(a,b)$, thus by Proposition 13 $d(a^v_\lambda, b^v_\lambda) = d(a', v) < d(a,b)$.

Case II. $a^v_\lambda \in \overrightarrow{bv}$ and $b^v_\lambda \notin \overrightarrow{av}$.

The justification in this case is analogous to the one in Case I.

Case III. $a^v_\lambda \notin \overrightarrow{bv}$ and $b^v_\lambda \notin \overrightarrow{av}$.

In this case, note that $a^v_\lambda = a^z_\lambda$ and $b^v_\lambda = b^z_\lambda$, so by Proposition 15, $d(a^v_\lambda, b^v_\lambda) \leq d(a,b)$. □

The proof of the cases that are considered in the following proposition's demonstration is similar to those used to prove the previous proposition.

Proposition 17 *Let $a, b, v \in X$, $0 \leq \lambda \leq d(a,v)$ and $0 \leq \kappa \leq d(b,v)$. If $a \notin \overrightarrow{bv}$ and $b \notin \overrightarrow{av}$, then $d(a^v_\lambda, b^v_\kappa) \leq d(a,b) + |\lambda - \kappa|$.*

From the previous propositions we obtain the following corollary.

Corollary 18 *If $a, b, v \in X$, $0 \leq \lambda \leq d(a,v)$ and $0 \leq \kappa \leq d(b,v)$, then $d(a^v_\lambda, b^v_\kappa) \leq d(a,b) + |\lambda - \kappa|$.*

Theorem 19 *Let X be a tree. Then the map $r : F_2(X) \to F_1(X)$, given by $r(A) = \{PM(A)\}$ is $4-$Lipschitz.*

Proof Let $A, B \in F_2(X)$. Note that $H(A, r(A)) = \frac{\delta(A)}{2}$. For the proof of this theorem we consider three cases:

Case I. If $|A| = 1 = |B|$. Then $H(r(A), r(B)) = H(A,B)$.

Case II. If $|A| = 2$ and $|B| = 1$. Note that in this case $\frac{\delta(A)}{2} \leq H(A,B)$, thus $H(r(A), r(B)) = H(r(A), B) \leq H(r(A), A) + H(A,B) \leq \frac{\delta(A)}{2} + H(A,B) \leq 2H(A,B)$.

Case III. If $|A| = 2 = |B|$. We consider the following two subcases:

Subcase a. $H(A,B) \geq \frac{\delta(A)}{2}$.

In this case it is true that

$$\begin{aligned} H(r(A), r(B)) &\leq H(r(A), A) + H(A,B) + H(B, r(B)) \\ &= \frac{\delta(A)}{2} + H(A,B) + \frac{\delta(B)}{2}. \end{aligned}$$

Now we consider the following two situations:

i) If $\delta(B) \leq \delta(A)$, it is satisfied that

$$H(r(A), r(B)) \leq H(A,B) + \delta(A) \leq H(A,B) + 2H(A,B) = 3H(A,B)$$

ii) If $\delta(B) > \delta(A)$, then by Lemma 3 we get

$$\delta(B) \leq 2H(A,B) + \delta(A) \leq 4H(A,B),$$

and thus

$$H(r(A), r(B)) \leq \frac{\delta(A)}{2} + H(A,B) + \frac{\delta(B)}{2} \leq 4H(A,B).$$

Subcase b. $H(A,B) < \frac{\delta(A)}{2}$.

Let $A = \{a, c\}$ and $B = \{b, d\}$, suppose that $H(A,B) = d(a,b)$, then by Corollary 18 and Lemma 3 we obtain

$$\begin{aligned} H(r(A), r(B)) &= d(PM(A), PM(B)) = d(a^{c}_{\frac{\delta(A)}{2}}, b^{c}_{\frac{\delta(B)}{2}}) \\ &\leq d(a,b) + |\frac{\delta(A)}{2} - \frac{\delta(B)}{2}| \leq 2H(A,B). \end{aligned}$$

Therefore, r is a $4-$Lipschitz retraction. □

Now we will prove that $F_2(X)$ is a Lipschitz retract of $F_3(X)$, for which we require the following results.

Proposition 20 *Let $a, b, d, e \in X$. If $d(a,d) < \frac{d(a,b)}{2}$ and $d(b,e) < \frac{d(a,b)}{2}$ then $PM(\{d, e\}) \in \overrightarrow{ab}$.*

Proof For the proof of this proposition we consider three cases:

Case I. $d, e \in \overrightarrow{ab}$.

If this happens then $PM(\{de\}) \in \overrightarrow{de} \subset \overrightarrow{ab}$.

Case II. $d \in \overrightarrow{ab}$ but $e \notin \overrightarrow{ab}$, or $e \in \overrightarrow{ab}$ but $d \notin \overrightarrow{ab}$.

Without loss of generality, suppose that $d \in \overrightarrow{ab}$ but $e \notin \overrightarrow{ab}$. In this situation we consider the following two subcases:

Subcase a. $\overrightarrow{ab} \subset \overrightarrow{ae}$. Note that $PM(\{d, b\}) \in \overrightarrow{ab}$ and since $d(b,e) < d(b,d)$ then $PM(\{d, e\}) = PM(\{d, b\})^{e}_{\frac{d(b,e)}{2}} \in \overrightarrow{ab}$.

Subcase b. $\overrightarrow{ab} \not\subset \overrightarrow{ae}$. In this subcase we have that there exists $t \in \overrightarrow{ab}$ such that $d(b,e) = d(b,t) + d(t,e)$. Since $d(a,b) = d(a,d) + d(d,t) + d(t,b)$ and $d(b,e) = d(b,t) + d(t,e)$, then $d(d,t) - d(t,e) = d(a,b) - d(a,d) - d(b,e) > 0$ and so $d(d,t) > d(t,e)$. Since $d, t \in \overrightarrow{ab}$, then $PM(\{d, e\}) = PM(\{d, t\})^{e}_{\frac{d(t,e)}{2}} \in \overrightarrow{at} \subset \overrightarrow{ab}$.

Case III. $d, e \notin \overrightarrow{ab}$. We consider the following two subcases:

Subcase a. $\overrightarrow{ab} \subset \overrightarrow{de}$. By Case II. $PM(\{d, b\}) \in \overrightarrow{ab}$ and $d(e,b) < d(a,b) \leq d(d,b)$, then $PM(\{d, e\}) = PM(\{d, b\})^{e}_{\frac{d(b,e)}{2}} \in \overrightarrow{ab}$.

Subcase b. $\overrightarrow{ab} \not\subset \overrightarrow{de}$. Let s, t such that $d(a,d) = d(a,s) + d(s,d)$ and $d(b,e) = d(b,t) + d(t,e)$.

i) If $\overrightarrow{st} \subset \overrightarrow{ab}$. Since $\overrightarrow{st} \subset \overrightarrow{de}$, applying the Subcase a. $PM(\{d, e\}) \in \overrightarrow{st} \subset \overrightarrow{ab}$.

ii) If $\overrightarrow{st} \not\subset \overrightarrow{ab}$ and $\overrightarrow{ab} \not\subset \overrightarrow{st}$. In this situation we have that $t \in \overrightarrow{ab}$ or $s \in \overrightarrow{ab}$, without loss of generality suppose that $t \in \overrightarrow{ab}$. Applying Case II. we have that $PM(\{d, t\}) \in \overrightarrow{at}$ and since $d(t,e) < d(t,d)$ then $PM(\{d, e\}) = PM(\{d, t\})^{e}_{\frac{d(t,e)}{2}} \in \overrightarrow{at} \subset \overrightarrow{ab}$. □

Proposition 21 *Let $x, y, z \in X$ be three different points. Then $|\overrightarrow{xy} \cap \overrightarrow{yz} \cap \overrightarrow{xy}| = 1$.*

Proof Since X does not have simple closed curves we have that the two following things are satisfied:

$\overrightarrow{xy} \cap \overrightarrow{zx}$ is $\{x\}$ or an arc contained in $\overrightarrow{xy}$, and
$\overrightarrow{xy} \cap \overrightarrow{zy}$ is $\{y\}$ or an arc contained in $\overrightarrow{xy}$.

Notice that $\overrightarrow{xy} \cap \overrightarrow{zx} = \{x\}$ and $\overrightarrow{xy} \cap \overrightarrow{zy} = \{y\}$ cannot happen otherwise we would have a simple closed curve.

- If $\overrightarrow{xy} \cap \overrightarrow{zx} = \{x\}$ and $\overrightarrow{xy} \cap \overrightarrow{zy}$ is an arc contained in $\overrightarrow{xy}$ then $\overrightarrow{xy} \cap \overrightarrow{yz} \cap \overrightarrow{xy} = \{x\}$.
- If $\overrightarrow{xy} \cap \overrightarrow{zx}$ is an arc contained in $\overrightarrow{xy}$ and $\overrightarrow{xy} \cap \overrightarrow{zy} = \{y\}$ then $\overrightarrow{xy} \cap \overrightarrow{yz} \cap \overrightarrow{xy} = \{y\}$.

- If $\overrightarrow{xy} \cap \overrightarrow{zx}$ is an arc contained in $\overrightarrow{xy}$ let's say $\overrightarrow{vx}$ and $\overrightarrow{xy} \cap \overrightarrow{zy}$ is an arc contained in $\overrightarrow{xy}$ let's say $\overrightarrow{wy}$, since X has no simple closed curves then $v = w$ and thus $\overrightarrow{xy} \cap \overrightarrow{yz} \cap \overrightarrow{xy} = \{v\}$. ☐

Proposition 22 *Let $x, y, z \in X$ be three different points. If $d(x,y) \le d(x,z)$ and $d(x,y) \le d(y,z)$ then $z^{PM(\{x,y\})}_{\frac{d(x,y)}{2}} \in \overrightarrow{zv}$, where v is the point that guarantees the previous proposition.*

Proof Let $z' = z^{PM(\{x,y\})}_{\frac{d(x,y)}{2}}$. For the proof of this proposition, we consider the following two cases: **Case I.** $v = x$ or $v = y$. Without loss of generality, suppose that $v = x$. Notice that $z \notin \overrightarrow{xy}$ and since $d(z,z') < d(z,x) < d(z, PM(\{x,y\}))$, then $z' \in \overrightarrow{zx}$.

Case II. If $v \ne x$ and $v \ne y$. In this case we note that $d(z,z') = \frac{d(x,y)}{2} \le \max\{d(x,v), d(v,y)\} \le d(v,z)$ and therefore $z' \in \overrightarrow{zv}$. ☐

Theorem 23 *Let X be a tree and $r : F_3(X) \to F_2(X)$ given by*

$$r(A) = \begin{cases} \left\{ PM(\{a,b\}), c^{PM(\{a,b\})}_{\frac{\delta(A)}{2}} \right\} & \text{if } |A| = 3, \\ A & \text{if } |A| < 3, \end{cases} \tag{7}$$

where $A = \{a,b,c\}$ and a, b are such that $\delta(A) = d(a,b)$. Then r is 4-Lipschitz.

Proof Let $A, B \in F_3(X)$. Notice that $H(A, r(A)) = \frac{\delta(A)}{2}$. We consider the following three cases:

Case I. If $|A| < 3$ and $|B| < 3$, then $H(r(A), r(B)) = H(A,B)$.

Case II. If $|A| = 3$ and $|B| < 3$, notice that $\frac{\delta(A)}{2} \le H(A,B)$, then

$$H(r(A), r(B)) \le 2H(A,B),$$

its proof is identical to Case II. of Theorem 19.

Case III. If $|A| = 3 = |B|$. We consider the following two subcases:

Subcase a. If $H(A,B) \ge \frac{\delta(A)}{2}$ then $H(r(A), r(B)) \le 4H(A,B)$, the proof is identical to Subcase a. of Case III. of Theorem 19.

Subcase b. $H(A,B) < \frac{\delta(A)}{2}$. Suppose that $B = \{d,e,f\}$ such that $d(a,d) < \frac{\delta(A)}{2}$, $d(b,e) < \frac{\delta(A)}{2}$ and $d(c,f) < \frac{\delta(A)}{2}$. Notice that

$$H(A,B) = \max\{d(a,d), d(b,e), d(c,f)\}. \tag{8}$$

From here we have three situations, $\delta(B) = d(d,e)$, $\quad \delta(B) = d(e,f)$ or $\delta(B) = d(d,f)$.

If $\delta(B) = d(d,e)$. Let $c' = c^{PM(\{a,b\})}_{\frac{\delta(A)}{2}}$ and $f' = f^{PM(\{d,e\})}_{\frac{\delta(B)}{2}}$. Note that in this situation

$$H(r(A), r(B)) \le \max\{d(PM(\{a,b\}), PM(\{d,e\})), d(c',f')\}. \tag{9}$$

On the other hand, by Proposition 20 we have that $PM(\{d,e\}) \in \overrightarrow{ab}$ and thus $d(PM(\{a,b\}), PM(\{d,e\})) = d\left(a^b_{\frac{\delta(A)}{2}}, d^b_{\frac{\delta(b)}{2}}\right)$, then by (8), Corollary 18 and Lemma 3 we have that

$$d(PM(\{a,b\}),PM(\{d,e\}))\leq d(a,d)+\left|\frac{\delta(A)-\delta(B)}{2}\right|\leq 2H(A,B). \tag{10}$$

Now we will prove that $d(c',f')\leq 2H(A,B)$, for this purpose we consider the following two situations:

(a) $\overrightarrow{ab}\subset\overrightarrow{ac}$ or $\overrightarrow{ab}\subset\overrightarrow{bc}$. Without loss of generality suppose that the first happens, then by Corollary 18, (8) and Lemma 3 we have that

$$d(c',f')=d(c^a_{\frac{\delta(A)}{2}},f^a_{\frac{\delta(B)}{2}})\leq d(c,f)+\left|\frac{\delta(A)-\delta(B)}{2}\right|\leq 2H(A,B). \tag{11}$$

(b) $\overrightarrow{ab}\not\subset\overrightarrow{ac}$ and $\overrightarrow{ab}\not\subset\overrightarrow{bc}$. Let $v=\overrightarrow{ab}\cap\overrightarrow{bc}\cap\overrightarrow{ac}$. Since $PM(\{a,b\}),PM(\{d,e\})\in\overrightarrow{ab}$, then we have the following two options:

   i. $PM(\{a,b\}),PM(\{d,e\})\in\overrightarrow{va}$ or $PM(\{a,b\}),PM(\{d,e\})\in\overrightarrow{vb}$. Without loss of generality suppose that the first happens, then $d(c',f')=d(c^a_{\frac{\delta(A)}{2}},f^a_{\frac{\delta(B)}{2}})$, thus by Corollary 18, (8) and Lemma 3 we obtain

$$d(c',f')\leq d(c,f)+|\frac{\delta(A)-\delta(B)}{2}|\leq 2H(A,B). \tag{12}$$

   ii. $(PM(\{a,b\})\in\overrightarrow{av}$ and $PM(\{d,e\})\in\overrightarrow{vb})$ or $(PM(\{a,b\})\in\overrightarrow{vb}$ and $PM(\{d,e\})\in\overrightarrow{av})$. Without loss of generality suppose that the first happens. Note that in this situation $v\in\overrightarrow{be}$. Thus, by Proposition 22 we have that $c'\in\overrightarrow{cv}$ and $f'\in\overrightarrow{fv}$, which implies that $d(c',f')=d(c^v_{\frac{\delta(A)}{2}},f^v_{\frac{\delta(B)}{2}})$. By Corollary 18, (8) and Lemma 3 we have that

$$d(c',f')\leq d(c,f)+|\frac{\delta(A)-\delta(B)}{2}|\leq 2H(A,B). \tag{13}$$

Therefore, (9), (10), (11), (12) and (13) imply in this situation that $H(r(A),r(B))\leq 2H(A,B)$.

If $\delta(B)=d(e,f)$. Let $d'=d^{PM(\{e,f\})}_{\frac{\delta(B)}{2}}$ and c' be as in the previous situation. Notice that in this situation

$$H(r(A),r(B))\leq\max\{d(PM(\{a,b\}),d'),d(c',PM(\{e,f\})\} \tag{14}$$

By Proposition 20 we have that $PM(\{e,f\})\in\overrightarrow{bc}$. We consider the following two possibilities:

(a) $\overrightarrow{ab}\subset\overrightarrow{ac}$ or $\overrightarrow{ab}\subset\overrightarrow{bc}$. Without loss of generality, suppose that the first thing happens. Since $PM(\{e,f\})\in\overrightarrow{bc}$ we get

$$d(d',PM(\{a,b\}))=d(d^c_{\frac{\delta(B)}{2}},a^c_{\frac{\delta(A)}{2}}).$$

Using Corollary 18, (8) and Lemma 3 we obtain

$$d(d',PM(\{a,b\}))\leq d(a,b)+\left|\frac{\delta(A)-\delta(B)}{2}\right|\leq 2H(A,B). \tag{15}$$

On the other hand, since $c'\in\overrightarrow{cb}$ and $PM(\{e,f\})\in\overrightarrow{bc}$, then $d(c',PM(\{e,f\}))=d(c^b_{\frac{\delta(A)}{2}},f^b_{\frac{\delta(B)}{2}})$ by Corollary 18, (8) and Lemma 3 we have

$$d(c',PM(\{e,f\}))\leq d(c,f)+\left|\frac{\delta(A)-\delta(B)}{2}\right|\leq 2H(A,B). \tag{16}$$

(b) $\overrightarrow{ab}\not\subset\overrightarrow{ac}$ and $\overrightarrow{ab}\not\subset\overrightarrow{bc}$. Let $v=\overrightarrow{ab}\cap\overrightarrow{bc}\cap\overrightarrow{ac}$. By Proposition 22 we have that $c'\in\overrightarrow{cv}$ and since $PM(\{e,f\})\in\overrightarrow{bc}$ then $PM(\{e,f\})\in\overrightarrow{cv}$ or $PM(\{e,f\})\in\overrightarrow{vb}$. Hence, in any case $d(c',PM(\{e,f\}))=d(c^b_{\frac{\delta(A)}{2}},f^b_{\frac{\delta(B)}{2}})$, by Corollary 18, (8), and Lemma 3 we obtain

$$d(c',PM(\{e,f\}))\leq d(c,f)+\left|\frac{\delta(A)-\delta(B)}{2}\right|\leq 2H(A,B). \tag{17}$$

On the other hand, since $PM(\{a,b\})\in\overrightarrow{ab}$, $PM(\{e,f\})\in\overrightarrow{cb}$, then we have the following options:

$PM(\{a,b\})\in\overrightarrow{va}$ and $PM(\{e,f\})\in\overrightarrow{vc}$.
$PM(\{a,b\})\in\overrightarrow{va}$ and $PM(\{e,f\})\in\overrightarrow{vb}$.

$PM(\{a,b\}) \in \overrightarrow{vb}$ and $PM(\{e,f\}) \in \overrightarrow{vc}$.
$PM(\{a,b\}) \in \overrightarrow{vb}$ and $PM(\{e,f\}) \in \overrightarrow{vb}$.

Notice that in the four options $d(PM(\{a,b\}),d') = d(a^b_{\frac{\delta(A)}{2}}, d^b_{\frac{\delta(B)}{2}})$. Thus, using Corollary 18, (8) and Lemma 3 we have that

$$d(PM(\{a,b\}),d') \leq d(a,b) + \left|\frac{\delta(A)-\delta(B)}{2}\right| \leq 2H(A,B). \tag{18}$$

Therefore, (14), (15), (16), (17) and (18) imply in this situation that $H(r(A),r(B)) \leq 2H(A,B)$.

If $\delta(B) = d(d,f)$. Observe that in this situation only $\overrightarrow{ab} \not\subset \overrightarrow{ac}$ and $\overrightarrow{ab} \not\subset \overrightarrow{bc}$ are satisfied, where it is true that $H(r(A),r(B)) \leq 2H(A,B)$, which proof is analogous to that of possibility (b) from 2.

Therefore, from the Subcase b, we obtain that $H(r(A),r(B)) \leq 2H(A,B)$. Concluding, thus, the proof of theorem. □

Note that theorems 19 and 23 improve the Lipschitz constant for the case of m-ods.

Corollary 24 *If X is a tree, then $F_1(X)$ is a* 16-*Lipschitz retract of* $F_3(X)$.

Recently in [2, Corollary 5.2, p. 8] it is shown that Lipschitz constants must depend on n whenever X is a metric space that admits Lipschitz retraction onto a line segment. Note that in a m -od or even a tree these conditions are satisfied, so we have the following problem.

Problem 25 Improve the Lipschitz constant of theorems 10, 19 and 23, as well as corollaries 11 and 24.

On the other hand, the main result of [3] guarantees the existence of a Lipschitz retraction of $F_n(X)$ onto $F_{n-1}(X)$ for $n \geq 2$, when X is an Hadamard space, in particular for trees with path metric. So, in relation to theorems 19 and 23, the following problem is interesting

Problem 26 Give an explicit Lipchitz retraction $r : F_n(X) \rightarrow F_k(X)$ (i.e, that $r(A)$ only depends on A, for every $A \in F_n(X)$), where X a tree, $n \geq 4$ and $n > k \geq 1$.

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